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The Differential Geometry and Physical Basis for the Application of Feynman Diagrams by Samuel L. Marateck

QED, differential geometry, geometry in physics, magnetic vector potential, Maxwell's equations, Aharonov-Bohm effect, gauge theory, Yang Mills Theory, Feynman diagrams

The design of a commemorative stamp tells a wonderful story. The Feynman diagrams on it show how Feynman's work, originally applicable to QED, for which he won the Nobel Prize, was then later used to elucidate the electroweak force. The design is meaningful to both mathematicians and physicists. For mathematicians, it demonstrates the application of differential geometry. For physicists, it depicts the verification of QED; the application of the Yang-Mills equations; and the establishment and experimental verification of the electroweak force, the first step in the creation of the standard model. The physicists used gauge theory to achieve this and were for the most part unaware of the developments in differential geometry. Similarly, mathematicians developed fiber bundle theory without knowing that it could be applied to physics. We should, however, remember that in general relativity, Einstein introduced geometry into physics. And as we will relate below, Weyl did so for electromagnetism.

General relativity sparked mathematicians' interest in parallel transport, eventually leading to the development of fiber bundles in differential geometry. After physicists achieved success using gauge theory, mathematicians applied it to differential geometry. The story begins with Maxwell's equations. In this story the vector potential A goes from being a mathematical construct used to facilitate problem solution in electromagnetism to taking center stage by causing the shift in the interference pattern in the Aharonov-Bohm solenoid effect. As the generalized four-vector A_μ , it becomes the gauge field that mediates the electromagnetic interaction and the electroweak and strong interactions in the standard model of physics; A_μ is understood as the connection on fiber-bundles in differential geometry.

Maxwell's Equations

Since the curl of a vector can not exist in two dimensions, Maxwell's equations suggest that light as we know it, cannot exist in a 2-D world. This is the first clue that electromagnetism is bound up with geometry.

Gauge Invariance

In a 1918 article Hermann Weyl tried to combine electromagnetism and gravity by requiring the theory to be invariant under a local scale change.

Although this article was unsuccessful, it did introduce the concept of "gauge invariance"; his term was Eichinvarianz. His paper also introduced the geometric interpretation of electromagnetism and the beginnings of nonabelian gauge theory. The similarity of Weyl's theory to nonabelian gauge theory is more striking in his 1929 paper. By 1929 Maxwell's equations had been combined

google nonabelian gauge theory,

<http://encyclopedia2.thefreedictionary.com/Non-abelian+gauge+theories>

fiber bundle

abelian gauge theory,

gauge field

with quantum mechanics to produce the start of quantum electrodynamics. In his 1929 article Weyl turned from trying to unify electromagnetism and gravity to introducing as a phase factor an exponential in which the phase α is preceded by the imaginary unit i , e.g., $e^{+iq\alpha(x)}$, in the

wave function for the wave equations (for instance, the Dirac equation is $(i\gamma^\mu\partial_\mu - m)\psi = 0$). It is here that Weyl correctly formulated gauge theory as a symmetry principle from which electromagnetism could be derived. It was to become the driving force in the development of quantum field theory. Weyl's gauge theory appears to be the basis of Yang-Mill's gauge theory.

Thus gauge invariance determines the type of interaction; here, the inclusion of the vector potential. This is called the gauge principle, and A_μ is called the gauge field or gauge potential. Gauge invariance is also called gauge symmetry. In electromagnetism A is the space-time vector potential representing the photon field, while in electroweak theory A represents the intermediate vector bosons W^\pm and Z^0 fields; in the strong interaction, A represents the colored gluon fields. The fact that the q in $\psi(x)e^{+iq\alpha(x)}$ must be the same as the q in $\partial_\mu + iqA_\mu$ to insure gauge invariance means that the charge q must be conserved.

Thus gauge invariance dictates charge conservation. By Noether's theorem, a conserved current is associated with a symmetry. Here the symmetry is the nonphysical rotation invariance in an internal space called a fiber. In electromagnetism the rotations form the group $U(1)$, the group of unitary 1-dimensional matrices. $U(1)$ is an example of a gauge group, and the fiber is S^1 , the circle.

For the wave equations to be gauge invariant, i.e., have the same form after the gauge transformation as before, the local phase transformation $\psi(x) \rightarrow \psi(x)e^{+iq\alpha(x)}$ has to be accompanied by the local gauge transformation

$$(9) A_\mu \rightarrow A_\mu - q^{-1}\partial_\mu\alpha(x).$$

A fiber bundle is determined by two manifolds and the structure group G which acts on the fiber: the first manifold, called the total space E , consists of many copies of the fiber F —one for each point in the second manifold, the base manifold M which for our discussion is the space-time manifold. The fibers are said to project down to the base manifold. A principal fiber bundle is a fiber bundle in which the structure group G acts on the total space E in such a way that each fiber is mapped onto itself

and the action of an individual fiber looks like the action of the structure group on itself by left translation. In particular, the fiber F is diffeomorphic to the structure group G . The gauge principle shows how electromagnetism can be introduced into quantum mechanics.

In his 1929 paper Weyl also includes an expression for the curvature Ω of the connection A , namely, Cartan's second structural equation, which in modern differential geometry notation is $\Omega = dA + A \wedge A$. It is the same form as the equation used by Yang and Mills, which in modern notation is $\Omega = dA + 1/2 [A,A]$.

Since the transformations in (9) form an abelian group $U(1)$, the space-time vector potential A commutes with itself. Thus in electromagnetism the curvature of the connection A is just

$$(10) \Omega = dA$$

Differential Geometry

Differential geometry—principally developed by Levi-Civita, Cartan, Poincaré, de Rham, Whitney, Hodge, Chern, Steenrod and Ehresmann—led to the development of fiber bundle theory, which is used in explaining the geometric content of Maxwell's equations. It was later used to explain Yang-Mills theory and to develop string theory.

In 1959 Aharonov and Bohm established the primacy of the vector potential by proposing an electron diffraction experiment to demonstrate a quantum mechanical effect: A long solenoid lies behind a wall with two slits and is positioned between the slits and parallel to them. An electron source in front of the wall emits electrons that follow two paths: one path through the upper slit and the other path through the lower slit. The first electron path flows above the solenoid, and the other path flows below it. The solenoid is small enough so that when no current flows through it, the solenoid does not interfere with the electrons' flow. The two paths converge and form a diffraction pattern on a screen behind the solenoid. When the current is turned on, there is no magnetic or electric field outside the solenoid, so the electrons cannot be affected by these fields. However, there is a vector potential A , and it affects the interference pattern on the screen.

Thus Einstein's objection to Weyl's 1918 paper can be understood as saying that there is no Aharonov-Bohm effect for gravity.

Wu and Yang analyzed the Aharonov Bohm prediction and commented that different phase shifts $(q/\hbar c)\phi$ may describe the same interference pattern, whereas the phase factor $e^{(iq/\hbar c)\phi}$ provides a unique description as we now show. (\hbar = Planck's constant)

Let's see how by using the primacy of the four vector potential A , we can derive the homogeneous Maxwell's equations from differential geometry simply by using the gauge transformation. Then we will get the nonhomogeneous Maxwell's equations using the fact that our world is a four-dimensional (space-time) world.

In electromagnetic theory, two fundamental principles are $\nabla \cdot B = 0$ (no magnetic monopoles) and for time-independent fields $E = -\nabla A_0$ (the electromagnetic field is the gradient of the scalar potential), so consistency dictates that in the time-dependent case, we assign the two terms to B and E respectively:

$$(11) B = \nabla \times A \text{ and } E = -\nabla A_0 - \partial_t A .$$

The gradient, curl, and divergence are spatial operators—they involve the differentials dx , dy , and dz . The exterior derivative of a scalar is the gradient, the exterior derivative of a spatial 1-form is the curl, and the exterior derivative of a spatial twoform is the divergence. In the 1-form A , the $-A_0 dt$

google spatial 1-form,

http://www.physicsinsights.org/pbp_one_forms.html

spatial twoform

is a spatial scalar and when the exterior derivative is applied gives rise to ∇A_0 . The remaining terms in A are the coefficients of dx^i constituting a spatial 1-form and thus produce $\nabla \times A$. We define the field strength F as $F = dA$, and from equation (10) we see that the field strength is the curvature of the connection A .

Feynman Diagrams

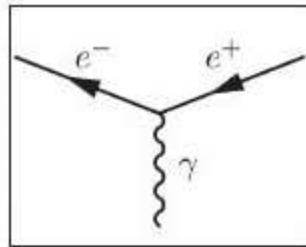


Figure 1. A pair production vertex.

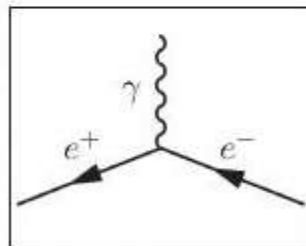


Figure 2. A pair annihilation vertex.

Feynman introduced schematic diagrams, today called Feynman diagrams, to facilitate calculations of particle interaction parameters. External particles, represented by lines (edges) connected to only one vertex are real, i.e., observable. They are said to be on the mass shell, meaning their four-momentum squared equals their actual mass, i.e., $m^2 = E^2 - p^2$. Internal particles are represented by lines that connect vertices and are therefore intermediate states—that is why they are said to mediate the interaction. They are virtual and are considered to be off the mass shell. This means their

four-momentum squared differs from the value of their actual mass.

(Figure 1) is a vertex diagram and as such represents a component of a Feynman diagram. It illustrates the creation of an electron-positron pair from a photon, γ ; it is called pair production. The γ is represented by a wavy line. The Feynman-Stueckelberg interpretation of negative-energy solutions indicates that here the positron, the electron's antiparticle, which is propagating forward in time, is in all ways equivalent to an electron going backwards in time.

If all the particles here were external, the process would not conserve energy and momentum. To see this you must first remember that since the photon has zero mass due to the gauge invariance of electromagnetic theory, its energy and momentum are equal. Thus β , which equals p/E (momentum/energy), has the value 1; but $\beta = vc$, so that the photon's velocity is always c , the speed of light.

In the diagram, the electron and positron momenta are equal and are in opposite directions. The photon travels at the speed of light, and therefore its momentum cannot be zero; but there is no particle to cancel its momentum, so the interaction cannot occur. (For it to occur requires a Coulomb field from a nearby nucleus to provide a virtual photon that transfers momentum, producing a nuclear recoil.) Therefore the γ in the diagram is internal. Its mass is off the mass shell and cannot equal its normal value, i.e., zero.

(Figure 2) is also a vertex diagram and represents an electron-positron pair annihilation producing a γ . Again, if all the particles are external, conservation of energy and momentum prohibit the reaction from occurring; therefore the γ must be virtual.