Information Entropy and Correlations in Prime Numbers

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Abstract

The difference between two consecutive prime numbers is called the distance between the primes. We study the statistical properties of the distances and their increments (the difference between two consecutive distances) for a sequence comprising the first 5×10^7 prime numbers. We find that the histogram of the increments follows an exponential distribution with superposed periodic behavior of period three, similar to previously-reported period six oscillations for the distances.

Recent reports indicate that many physical and biological systems exhibit patterns where prime numbers play an important role. Examples range from the periodic orbits of a system in quantum chaos to the life cycles of species [1, 2, 3, 4, 5, 6, 7]. Recent work reports on a potential for which the quantum energy levels of a particle can be mapped onto the sequence of primes [8]. Furthermore, it has been shown that a gas of independent bosons with energies equal to the logarithm of consecutive primes possesses a canonical partition function coinciding with the Riemann function [9]. The partition function of a system with energies equal to the distances between two consecutive prime numbers behaves like a set of non-interacting harmonic oscillators [10]. Most recently, power-law behavior in the distribution of primes and correlations in prime numbers have been found [11], along with multifractal features in the distances between consecutive primes [12]. Previous work thus further motivates studies of prime numbers using methods of statistical physics. Here, we focus on the statistical properties of the distances between consecutive prime numbers and the increments in these distances [Fig. 1].

Since the distribution of distances is well-studied, we discuss the occurrence frequency of increments between consecutive distances. We find that the distribution of increments [Fig. 2(a)] exhibits large peaks for given values of the increments and medium and small peaks for other values, and that these peaks follow period-three oscillation. Specifically, we find that the increments with values of $6k + 2$ ($k = 0, 1, 2, 3...$) have the highest occurrence frequency, followed by increments with values of $6k + 4$. Values of $6k$ are relatively rare and correspond to the small peaks in the distribution. This regularity is present for both positive and negative increments and does not depend on the length N_p of the sequence. We also find that the occurrence frequency of increments decreases exponentially and that this exponential behavior is well pronounced for both large and small peaks, forming a "double-tent" shape [Fig. 2(b)].

We find exponential behavior with superposed periodic behavior with period-three oscillation for the distribution of increments similar to the period-six oscillation for the distribution of distances [10]. Further, we find that the occurrence frequency of a positive increment is almost the same as the occurrence frequency of its negative counterpart for a given sequence length N_p [Fig. 2(c)].

In summary, we find a new statistical feature in the sequence of increments between consecutive prime distances. We find a period-three oscillation in the distribution of increments and this distribution follows an exponential form.

This empirical observation may be of importance in further understanding the nature of prime numbers as well as those physical and biological processes where prime numbers play a role.

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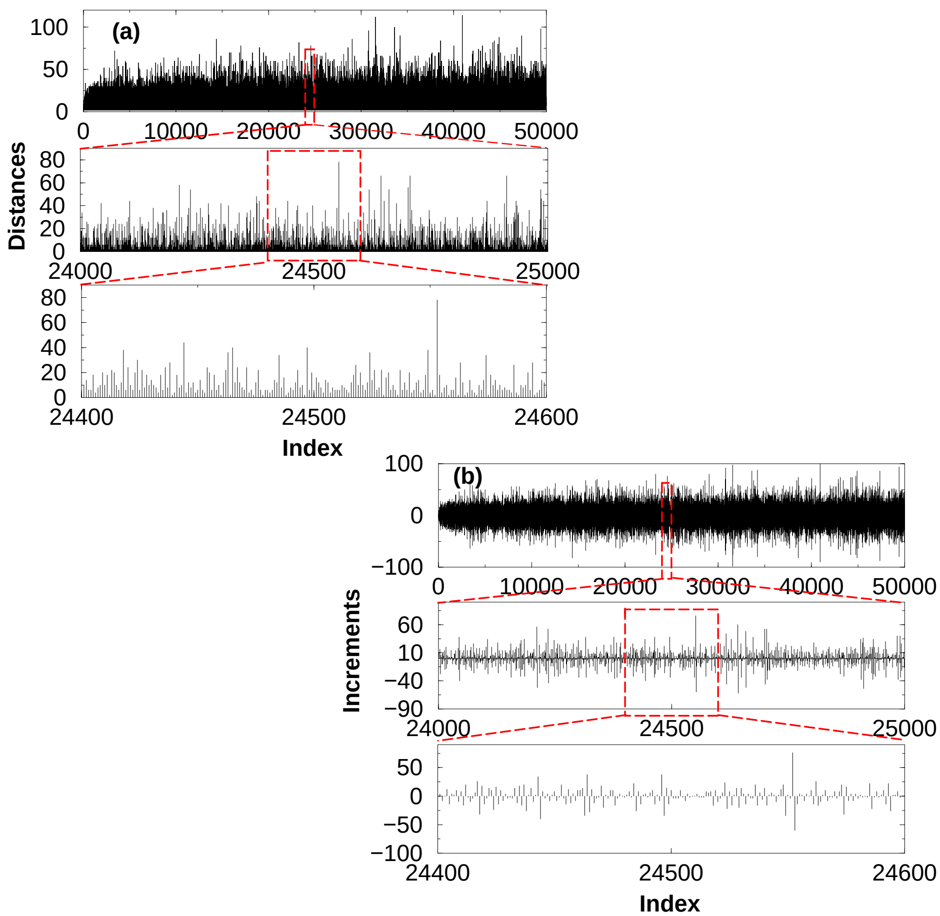

FIG. 1: Distances between consecutive prime numbers (indexed sequentially) and their increments. (a) The first 5×10^4 distances between consecutive prime numbers. (b) The first 5×10^4 increments.

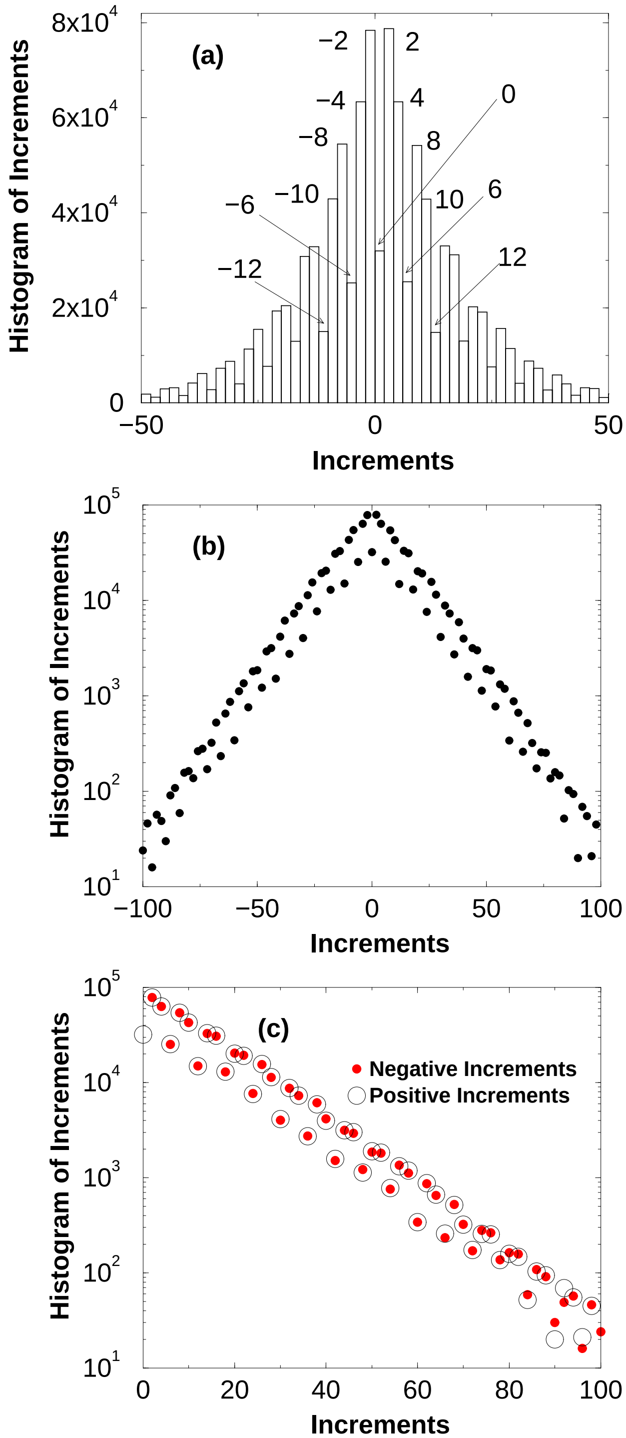

FIG. 2: (a) Histogram of increments in the distances between consecutive prime numbers for the sequence of the first $N_p = 10^6$ primes. The bin width is 1. The occurrence frequency of increments with given values exhibits a robust period-three oscillation. Increments with values $\pm(6k+2)$ $(k = 0, 1, 2, 3, ...)$ occur most often, increments with values $\pm(6k+4)$ occur less often, and increments with values $\pm 6k$ are rare. This regularity is always present regardless of the sequence length N_p. (b) Tent-shape of the histogram of increments on a linear-log plot suggests an exponential form for large, medium, and small peaks. The top curve (corresponding to large