Prime Number Conspiracy: The Biggest ideas in Math From Quanta Edited by Thomas Lin

Foreword James Gleick pg Xiii- XV

Prime numbers are those integers divisible only by themselves and 1.

Euclid proved there are infinitely many of them.

If you can talk about primes, you can also talk about gaps in primes. Often the gap is two, as in 11 and 13. They are twin primes. The twin primes conjecture says there are an infinite number of twin primes.

Sieves, a method for finding primes by filtering out non-primes. Sieves go back to Eratosthenes, but the algorithms are improving. Sieves led to combs, which have teeth. New terms include Groups, Motives, Weights, Graphs, Matroids, Doughnuts.

A new language, Robbert Dijkgraff suggests, comes from quantum theory. He proposes that mathematicians can learn from quantum theory, because, Dijkgraff suggests, quantum theory has a holistic approach, where the study of categories of objects focuses much more on mutual relations than on any specific example.

Erica Klarreich: Unheralded Mathematician Bridges the Prime Gap

Because prime numbers are fundamentally connected with multiplication, understanding their additive properties can be tricky.

A major effort is underway to bridge the gap between GPY and the bounded prime gap conjecture.

Prime numbers are abundant at the beginning of the number line, but grow much sparser among large numbers. Twin primes often show up, that differ by only two; like 11 and 12. This happens even for very large numbers. For hundreds of years, it has been speculated that there are an infinite number of twin primes. This idea was expanded to the conjecture that there should be infinitely many prime pairs for any possible prime gap, not just 2. A lone researcher, Yitang Zhang, showed that there is some number N smaller than 70 million, such that there are infinitely many pairs of primes that differ by N. other researchers have been working to reduce that bound.

Erica Klarreich: Mathematicians Discover Prime Conspiracy

Prime numbers repel other would-be primes that end in the same digit, and have varied predilections for being followed by primes ending in the other possible digits. Yet this does not upend the notion that primes behave randomly so much as point to how subtle their particular mix of randomness and order is.

Prime numbers are not random; they are completely determined. Yet in many ways they seem to behave like a list of random numbers, with one rule: the approximate density of primes near any number is inversely proportional to how many digits the number has.

Erica KlarreichMathematicians Chase Moonshine's Shadow

mtn John McKay had been studying the different ways of representing the structure of the "monster group", a huge algebraic object that might capture a new type of symmetry. If it existed, it would act in special ways in certain dimensions, the first two of which were 1 and 196883. In 1978, McKay happened to notice that the J-function, one of the most fundamental objects in number theory had as its first important coefficient 196884, the sum of the monster 's first two special dimensions. Most mtns though this was a fluke, since there was no reason to think the two mth objects were even remotely related. It was then discovered that the j-function's second coefficient is the sum of the first three special dimensions of the monster. Other mtns so many of these numerical relationships that they were no longer thought of as coincidences. In 1979, the paper "Monstrous Moonshine" conjectured that these relationships must result from some deep connection between the monster group and j-function.

In 1992, Richard Borcherds proved there was a bridge between the two distant realms of mathematics: string theory. His discovery touched off a revolution in pure math, leading to a new field known as generalized Kac Moody algebras. But from a string theory perspective, it was something of a backwater. The 24-D string theory that linked the monster and j-function was no longer considered interesting by string theorists. Monstrous Moonshine has turned out to be just the start of the story.

I 2015, a paper was posted on arxiv.org presenting a numerical proof of the Umbral Moonshine Conjecture, which proposes that in addition to monstrous moonshine, there are 23 other moonshines: correspondences between the members of a symmetry group and the coefficients of a special function on the other. The 23 new moonshines appear to be intertwined with some of the most central structures in string theory, 4-D objects known as K3 surfaces. This suggests that each of the 23 new moonshines, there must be a corresponding string theory model. "… When we understand what moonshine is, it will be in terms of physics." a researcher says.

The symmetries of any given shape have a natural sort of arithmetic to them. For example, rotating a square 90 degrees and then flipping it horizontally is the same as flipping it along a diagonal. In the1800s, mtns realized they could distill this type of arithmetic into an algebraic entity called a group. The same abstract group can represent the symmetries of many different shapes. In the 1900s, mtns worked to classify all possible groups. They found that while most simple finite groups fell into natural categories, there were 26 oddballs that defied categorization. Of these, the biggest, and last to be discovered, was the monster.

The J function belongs to a special class of functions whose graphs have repeating patterns similar to M. C. Escher's disc with angels and devils, which shrink ever smaller as they approach the boundary.

These “modular functions” are the heroes of number theory, playing a role in Andrew Wiles’ 1994 proof of Fermat’s last theorem.

As with a sound wave, the J-function’s repeating pattern can be broken down into a collection of pure tones, so to speak, with coefficients indicating how “loud” each tone is. It is in these coefficients that McKay found the link to the monster group. In the 1990s, building on earlier work, Borcherds showed that there is a certain string theory model in which the J-function and the monster group both play roles. The coefficients of the j-function count the ways strings can oscillate at each energy level, and the monster group captures the model’s symmetry at those energy levels. This gave mtns a way to study the monster group using the J-function, whose coefficients are easy to calculate.

For decades, Physicists have struggled to find a string theory geometry that produces the physical effects we see in the real world.

The most promising candidates for such a geometry include 4-D shapes known as K3 surfaces. Studying a more limited function researchers found coefficients that were the same as the special dimensions of another oddball group, the Mathieu 24 group, with nearly 250 million elements: a new moonshine.

It was found that the full moonshine picture lay in the writings of Srinivasa Ramanugen, who, as he lay dying wrote G.H. Hardy a second letter saying that he had discovered what he called "mock theta" functions, listing 17 examples of these functions, but did not explain what they had in common.   Eight decades later, Sander Zwegers found in 2002 that they all are examples of what came to be known as mock modular forms. Researchers finally found that M24 moonshine is one of 23 different moonshines, each making a connection between the special dimensions of a group and the coefficients of a mock modular form. Just as monstrous moonshine made connection between the monster group and the j-function, It is conjectured that for each of these moonshines there is a string theory in which the mock modular form counts the string states and the group captures the models symmetry. A mock modular form always has an associated modular function called its "shadow", so they named this conjecture the Umbral Moonshine Conjecture.

Borcherd's earlier proof of monstrous moonshine also builds on the work of Rananujan. The algebraic objects at the core of the proof were discovered in the 3 formulas that had so startled Hardy in Ramanugen's first letter. A numerical proof has been provided for the Umbral Moonshine Conjecture.

The string theory underlying Umbral Moonshine will probably be an important physical theory.

Physicists are also excited about a possible connection between Moonshine and quantum gravity.

Natalie Wolchover: In mysterious pattern, math and nature converge

In a decentralized bus system in Mexico, each bus driver paid a bystander to record when the bus ahead of him had departed the stop. If it had left recently, he would slow down; if it had ledt long ago he sped up. This maximized the profits for drivers. Czech physicist Petr Seba collected this data, and plotted thousands of bus departure times on a computer. They found that the interaction between drivers caused the spacing between bus departures to exhibit a distinctive pattern previously observed in quantum physics experiments. The same pattern has turned up in other unrelated settings. Scientists now believe th widespread phenomena, called "universality", stems from an underlying connection to math, and its helping them to model complex systems like the internet and the earth's climate.

The pattern was discovered in the 1950s in the energy spectrum of the uranium nucleus. It has also shown up in spectral measurements of composite materials.

“Universality” is a precise balance between randomness and regularity, where neighboring line repel each other so the result is more uniform than random. In the “Universality” spectrum of say, line spaceings, a math formula, called the “correlation function” gives the exact probability of finding two lines spaced a given distance apart. This pattern also appears in a purely mathematical setting: it defines the spacing between the eigenvalues, or solutions, of a vast matrix filled with random numbers.

“why so many physical systems behave like random matrices is still a mystery” said one researcher, Horng-Tzer Yau. Yau and others have characterized many new types of random matrices, which can conform to a variety of numerical distributions and symmetry rules. Regardless of their specific characteristics, the random matrices are found to exhibit that same pattern.

Eugene Wigner discovered universality in atomic spectra. The more complex a system is, the more robust its universality should be. P. 39

Whenever a system exhibits universality, the system is complex, and correlated enough to be treated like a random matrix, which means you can use a random matrix to model it. You can compute other parameters of the matrix model and use them to predict that the system may behave like the model.

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a similar technique may lead to improvements in climate change modeling. The presence of Universality in features similar to the energy spectrum of a material indicates that its components are highly connected, and that it will therefore conduct fluids, electricity or heat. Conversely, the absence of universality may show that a material is sparse and acts as an insulator. Ken Golden a Mtn, and his student, Ben Murphy, use this distinction to predict heat transfer and fluid flow in sea ice, both at the microscopic level and through patchworks of arctic melt ponds spanning hundreds of miles.

The spectral measure of a mosaic of melt ponds, taken from a helicopter, or from a sample of sea ice from an ice core, instantly exposes the states of these systems. “Fluid flow through sea ice mediates very important processes that need to be understood to incorporate sea ice into climate modeling. Golden et al have found that the spectrum of dense healthy bone exhibits universality, while porous osteoporotic bone does not. The same underlying math describes systems with large or small component particles. P. 40.

Quantum systems, which are “natural”, are governed by the rules of matrices. Freeman Dyson helped develop random matrix theory in the 1960s. “Every quantum system is governed by a matrix representing the total energy of the system, and the eigenvalues of the matrix are the energy levels of the system.” P. 41

The matrices representing simple atoms, such as hydrogen or helium, have eigenvalues that can be calculated exactly, whose value corresponds with stunning accuracy to measured energy levels the of atoms. The matrices of more complex entities eg uranium nucleus, are much more complicated. This is why, according to Dyson, such nuclei can be represented by random matrices. The elements of the unknown nuclei matrix are so complex that they become washed out. So, the unknown matrix of the element can be represented by a matrix filled with random numbers, and so exhibits universality.

Another question is how this relates to prime numbers. The zeros of the Riemann zeta function exhibit universality. The zeros of this zeta function are closely tied to the distribution of prime numbers.

It may be that it is not a matrix that lies at the core of universality, but some deeper, as yet undiscovered math structure. P. 41

Natalie Wolchover: At the far ends of a new universal law p. 43

Systems of many interacting components – be they species, integers, or subatomic particles, kept producing the same statistical curve, known as the Tracy-Widom distribution, after its discoverers Craig Tracy and Harold Widom.

seems to be related to the Gaussian distribution, which represents the natural variation of independent random variables . Like the Gaussian, the T-W distribution exhibits “universality”: diverse microscopic effects give rise to the same collective macroscopic behavior. Universal laws like this allow researchers to accurately model complex systems whose inner workings they know little about.   Why do certain laws seem to emerge from complex systems, regardless of the underlying mechanisms driving those systems? P. 44

Researchers already knew that N eigenvalues of a random matrix tend to space along the real number line according to a distinct pattern, with the largest eigenvalue near sqroot(2N). T-W shows how the eigenvalues build up to the max, with lopsided exponential tails . The T-W peak occurs at the x axis value of sqroot(2N) where N is the number of variables in the system. sqroot(2N) corresponds to the largest eigenvalue of the matrix.

The peak of the T-W curve represents the transition point between stability and instability, such as that May calculated for his model ecosystem.

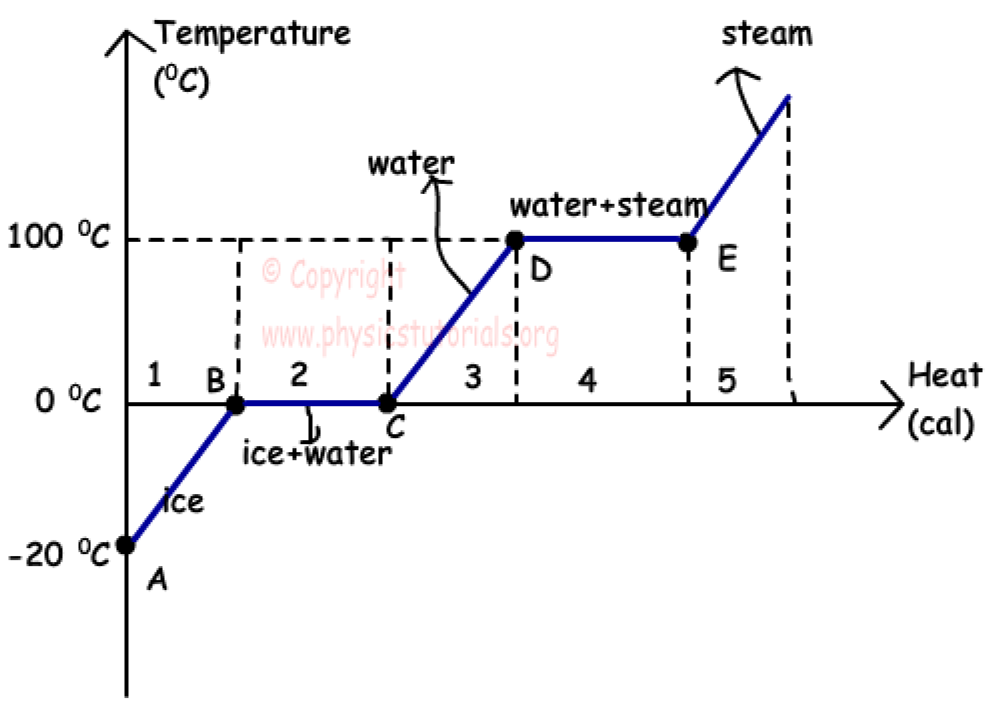
When T-W turned up in contexts that had nothing to do with random matrix theory, researchers began searching for the hidden thread that tied all the manifestations together.

The central limit theorem shows that uncorrelated variables form a bell curve, but highly correlated variables form a T-W curve. P. 46

Researchers studied the tails of the T-W curve to derive equations for them. They found that the right tail dropped off proportional to N, the number of eigenvalues, while the left hand tail dropped off proportional to N squared.

This form of the tails, sloping down from either side of a transitional peak, could be related to the universality of phase transitions; water freezing to ice; graphite becoming diamond; and metals changing into superconductors.

“Where the energy curve of water to steam has a kink (at D), the first derivative of the curve has a peak” This statement appears to contradict the image below.



Slope + constant slope zero

Similarly, the energy curves of certain strongly correlated systems have a kink at sqroot(2N). the associated peak for these systems in the T-W distribution, which appears in the third derivative of the energy curve. This makes the T-W distribution a third order phase transition. In 1980, theoretical physicists discovered a third-order phase transition separating a "weak coupling" phase, in which matter takes the form of nuclear particles, and a higher temperature "strong coupling" phase, in which matter becomes plasma. This was used to explain the phase transition from the "Big Bang" as it cooled.

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researchers have accrued evidence that the T-W distribution and its tails represent a universal phase transition between weak and strong coupling phases. In May's ecosystem model, the critical point square root(2N) separates a stable phase of weakly coupled species, whose populations can fluctuate individually without effecting the rest, from an unstable phase of strongly coupled species, in which fluctuations cascade through the system and throw it off balance; one phase in which components act alone, and one phase where components act in concert. P. 48

these are 3^{rd} order transitions

Natalie Wolchover: A Birds Eye view of nature's hidden order p. 51

Biologists found that the 5 different size cones in chicken's eyes have a haphazard yet uniform distribution. Avian vision has evolved over 300 million years, (as opposed to human vision, which has evolved over 30 million,) and is spectacularly successful.

A regular grid like spacing of cells is not optimal for different cell sizes. The biologists contacted the chemists, who found that the distribution of cones in the chicken eye was the same as the distribution of components in other organic and physical systems. This phenomena has been named hyperuniformity, and has turned up in quasi-crystals, matrices filled with random numbers, large scale structures of the universe, quantum collections, emulsions and colloids. P. 52.

The geometry of 5 different cone sized demands that they be disordered, but a competing evolutionary demand is for the retina to sample light as uniformly as possible; with blue, red, and green cones separated from their same colors as far as possible. P. 53

Hyperuniform systems fall into 2 classes: 1: quasicrystals: solids whose interlocked atoms appear to follow no repeating pattern, yet tessellate space. These settle to equilibrium. It is mutual repulsion between the atoms that space them apart and give hyperuniformity. 2: this non-equilibrium class is less well understood, and includes emulsions, colloids, and groups of cold atoms.   p. 54.

French physicist Denis Bartolo found that hyperuniformity could be induced in emulsions by mixing them at a critical amplitude so that they do not re-separate. Etaphase will make Hyperuniform photonic circuits; devises that transmit data via light rather than via electrons. It was discovered that Hyperuniform materials can have "band gaps", which block certain frequencies from propagating. Band gaps enable controlled transmission of data, since the blocked frequencies can be contained and guided through channels called wave guides. But band gaps were once thought to be unique to crystal lattices, and direction dependent, aligning with the crystal's symmetry axes. That meant photonic waveguides could only go in certain directions, limiting their use as circuits. Since Hyperuniform materials have no preferred direction, their band gaps are potentially more useful. P. 57.

Kevin Hartnett: A Unified Theory of Randomness p. 59

Researchers have found that random shapes can be categorized into various classes with distinct properties. Some kinds of random objects have clear connections with other kinds of random objects. This work forms the beginning of a unified theory of geometric randomness. P. 59

Certain probability measures on random geometries are special.

The first kind of random shape to be understood in this way was the random walk. P. 62

This type of motion was first discovered by Robert Brown who observed the random movements of pollen grains suspended in water. This seemingly random motion was caused by individual water molecules bumping into pollen grains. In the 1920s Norbert Wiener of MIT gave a precise math description of this process, now called Brownian motion, which is the "scaling limit" of random walks; as the length of step size and time between steps gets smaller, Brownian motion is the limit. This is the shape that almost all random walks converge to over time. These are 1-D curves.

2-D random spaces first preoccupied physicists as they tried to understand the structure of the universe. String theory posits 2-D curves, called worldsheets, and encodes the history of the 1-D string as it wiggles through time. "to make sense of quantum physics for strings, …you want to have something like Brownian motion for surfaces. In the 1980s, A. Polyakov described these surfaces that came to be called Liouville quantum gravity (LQG), an incomplete view of random 2-D surfaces. In parallel, another model, called the Brownian map, provided a different way to study these surfaces. Effort is now under way to see if these two models describe the same thing.it was decided to invent a way to measure distance on LQG surfaces, and then show that this distance measurement was the same as the distance measurement that came packaged with the Brownian map. Random non-crossing paths occur in nature. In 1999, Oded Schramm introduced the SLE curve, for Schramm-Loewner evolution as a canonical non-crossing random curve. SLE curves turned out to be important, because it was found that that you can construct random growth using SLEs, and there is a connection between SLEs and LQG. P. 69.

To Feynman diagrams physicists add numbers, for mass, momentum, and direction of the particles involved. These numbers are used to calculate a Feynman probability, the chance that the particle collision will play out as sketched. These diagrams have limitations: 1. Physicists are trying for increasingly high energy particle collisions that require greater precision of measurement, which means increased intricacy of the Feynman diagrams needed to produce a prediction. 2. Feynman diagrams are based on the assumption that the more potential collisions physicists account for, the more accurate their numerical predictions will be. This process of calculation, known as perturbative expansion. P. 73 expansion, works very well for collisions of electrons, where the EM and weak forces dominate. It works less well for high energy collisions like between protons within the nucleus, where the strong force prevails. In this case, accounting for a wider range of collisions, by drawing more complex Feynman diagrams, can actually diverge from real world physics.

However, over the last decade physicists and mtns have been exploring the strange fact that the values calculated from the Feynman diagrams seem to match some of the most important numbers that crop up in a branch of math known as algebraic geometry. These values are called “periods of motives”. P. 74.