Notes on *The Millennium Problems* Keith Devlin, Granta Publications, 2002

“The Millennium Problems” are Seven unsolved mathematical problems judged by the Clay Mathematics Institute in 2000 to be the greatest unsolved math problems of our time. Briefly, these problems are: The Riemann Hypothesis, Yang-Mills Theory and the Mas Gap Hypothesis, The P vs NP problem, The Navier Stokes Equations, the Poincare Conjecture, Knowing When the Equation Can’t Be Solved. And the Hodge Conjecture. The first five of these problems have direct consequences for physics.

One: The Music of the Primes

The Riemann Hypothesis p 19-62

Euclid found that every counting number bigger than 1 is either prime or has a unique prime decomposition. This fact, called the **fundamental theorem of arithmetic**, tells us that the primes are like the chemists atoms: the fundamental building blocks out of which all numbers are constructed. P. 20

In 1791, Gauss noticed that the prime density D_N = P(N)/N, where P(N) is the number of primes below N, is approximately 1/ln(N) he conjectured that by making N big enough, the density D_N could be made as close as you please to 1/ln(N). this was later proved and is known as the **prime number theorem**. P. 31

The question is, why is there a connection between the irregularly spaced points on the x-axis, and the smooth ln(N) curve? P. 32

Each branch of math is a connected whole, and there are link to many of the branches. P. 33

The real power of using calculus with complex numbers comes when you realize that you can have rules that associate not only a real number r=f(z) with a complex number z, but also rules that associate a second complex number w=f(z) with z. Such a rule is called a “complex valued function of a complex variable”, or a complex function.

Ie; one complex number z in complex plane, and a height r above complex plane; OR

one complex number z in complex plane, and a complex “height” w=f(z) above complex plane.

Although the “landscape” you get in this way cannot be visualized, the math: algebra, geometry and calculus still works, in many cases giving a much faster and easier result than you cn get with real numbers. This is exactly what happens with the pattern of the primes.

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Euler’s Zeta function can give us information about the prime numbers.

$$\zeta(n) = \sum_{j=1}^{\infty} \frac{1}{j^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots$$

$$\approx 1 + \frac{1}{(n-1)2^{n-1}} + \frac{1}{2^{n+1}}$$

Rearranging,

$$\boxed{\zeta(n) \approx 1 + \frac{n+3}{n-1}\frac{1}{2^{n+1}}}$$

Riemann found a close connection between the density function of prime numbers and the zeros of the complex Zeta function; that is, the solutions of the Zeta function = 0.

Riemann found that the zeros of the Zeta function occur whenever z is a negative even whole number. he also showed that the Zeta function has infinitely many other zeros in the complex plane, and hypothesized that all of those other complex zeros have the form z=1/2 + bi for some real number b. p. 45 this has not been proven yet, and is called the **Riemann Hypothesis**

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

The density function of prime numbers = DN = P(N)/N = the number of primes below N/N

In 1971, Hugh Montgomery found a formula that describes the spacing between the zeros of the Zeta function along the line x=1/2. Physicists immediately recognized the formula as similar to work in nuclear physics. In the 1980s, it was realized that Montgomery's formula gives the spacing between the energy levels of a **quantum chaotic system**. **Alain Conns** wrote down a system of equations that specifies a hypothetical quantum chaotic system that has all the prime numbers built in, and has proved that this system has energy levels corresponding to all the zeros of the Zeta function that lie on the critical line. If he can prove that there are no other zeros other than those corresponding to energy levels, he will have proved the **Riemann hypothesis.** p. 52

Two: The Fields We Are Made Of

Yang Mills Theory and the Mass Gap Hypothesis p. 63 -

Symmetry

Symmetries have “arithmetic”.

You can add together two symmetries of an object to get a third symmetry, and this addition has some of the similar properties of the ordinary addition of numbers. p. 88

These new kinds of arithmetic are called “groups”.

The set of all symmetries of an object is called the symmetry group for that object. The objects to which group theory can be applied don’t have to be physical objects. They can be abstract math objects, equations, or force fields. Many conservation laws arise from symmetries in the structure of the universe. Emmy Noether proved that every conservation law can be regarded as the result of some symmetry. P. 89

In 1918, mathematician Hermann Weyl set out to use the idea of symmetry to unify special relativity and electromagnetism. P. 89.

He picked invariance under a change in scale as the relevant symmetry, but this did not work. With the emergence of quantum theory, with its emphasis on wave function, it became clear that the crucial symmetry of the EM field is “gauge symmetry”, which means the field equations keep their form even if the EM potentials are multiplied by certain quantum mechanical phase factors, or gauges. This was the birth of “gauge theory”, in which the symmetry group assigned to each point in space time was called a gauge group. The important group for Maxwell’s equations were the one dimensional unity group U(1), which can be viewed as the set of rotations of the plane. Physicists were able to recast Maxwell’s theory as a gauge theory. P. 90.

The strategy for extending Maxwell’s theory to a quantum field theory incorporating one or more of the nuclear forces was to replace the gauge group U(1) with a more complicated symmetry group so that the resulting field could be quantum, and incorporate the other fundamental forces.

In the 1930s, Paul Dirac and others formulated the new theory of QED; a quantum description of electromagnetism. During the 1940s, Feynman, Julian Schwinger and others developed powerful methods for doing accurate calculations, making it the most precise scientific theory ever developed.

In 1954, Yang and Mills replaced the group U(1) by what is known as a “compact Lie Group” – a set of rigid motions of a complex multidimensional space. This would allow QED to be extended. U(1) is “abelian”; ie commutative; any two successive rotations can be done in any order. The compact Lie group however is non-abelian, which makes the math much trickier. P. 91-92.

The quantum version of the Yang-Mills equations, like Maxwell’s, describe zero mass waves that propagate at the speed of light. In quantum mechanics however, every particle can be considered a special kind of wave. The nuclear forces however, were known to be carried by particles of non-zero mass. This made quantum Yang-Mills equations problematic.

For the weak force, this problem was overcome in 1967 by Sheldon Glashow, Abdus Salam, and Steven Weinberg using a symmetry group SU(2) X U(1). They avoided the masslessness by introducing another force, the Higgs field. There is a mountain of evidence for this electro-weak theory. This theory also showed that at sufficiently high energy, the two forces merge into one, called the electroweak force.

Detection of the Higgs boson, which carries the Higgs field, would fill in the final piece of the puzzle..

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M P #2: “For any compact, simple gauge group, the quantum Yang-mills equations in 4-D Euclidian space, has a solution that predicts a mass gap.”

Nuclear forces known to be carried by particles of non-zero mass.

Discovery of a property of Y-M theory called asymptotic freedom, by David Gross and Frank Wilczekl

It says, roughly, that the interaction among quarks and gluons turns off at short distances, with quantum effects showing up only at a longer range. This not only explained some otherwise mysterious experimental results; it led to a unique quantum field theory incorporating the strong force, which became known as quantum chromodynamics. (QCD)

Many predictions of QCD have been confirmed experimentally. But our math understanding of the theory remains far from complete. No one has been able to solve the Yang-Mills equations. Instead, physicists use the equations to formulate rules for calculating various key numbers “approximately” to many decimal places.

When you think about it, it seems incredible. The most accurate scientific theory the world has ever seen is built upon equations that no one can solve.

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Wilczek: “the theory has to do the trick of producing massive particles from massless building blocks.”

Experiments, simulations, and calculations suggest there is a “mass gap”; ie, there is a non-zero minimum energy level for excitation of the vacuum. The mass gap property explains why the strong force has such short range.

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for all its origins in physics, the problem as stated is a math one. Indeed many physicists regard the problem in large part already solved. Wilczek comments: “one has direct evidence for the existence of… quarks and gluons, and for the basic interactions the theory postulates. Most of the evidence is from studies of jets in high energy processes, and comparison of their observed properties with very precise and unambiguous calculations in QCD, exploiting asymptotic freedom. Another sort of evidence comes from integrating the full equations using powerful computers. We not only know there is a mass gap, but also have calculated it, with results comparing favorably with reality…. Of course I understand that numerical results, however convincing and well controlled, are not traditionally considered mathematical proofs."

Yang Mills Theory and the Mass Gap Hypothesis was chosen to be a millennium problem because its solution would mark the beginning of a whole new era of mathematics. Ed Witten believes that finding a general solution to the Y-M equations "would essentially mean making sense of the standard model of particle physics…. Would shed light on a fundamental aspect of nature that physicists still do not properly understand."

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Note: no one has ever seen a quark; a book has been written titled Constructing Quarks by

in Emperors New Mind, Penrose calls QCD "useful", not superb.

A book slammed QCD; was it Emperors New Mind?

The key to the math study of symmetry is to look at transformations of objects rather than the objects themselves. To the Mtn, a symmetry transformation of a figure leaves the figure "invariant" or unchanged. P. 99

The math of symmetry is what is behind the proof of the fact that there are exactly 17 ways to repeat a particular local pattern.

The arithmetic of symmetry groups works in some ways like the arithmetic of numbers. The discovery of this new arithmetic in the later 18th century opened the door to new math results that effected not only math, but physics, chemistry, crystallography, medicine, engineering, communications and computer technology. P100

Symmetry arithmetic is similar to number arithmetic in the following 3 ways:

If S, T, and W are transformations of a symmetry group of say a circle, then the operation is associative, as in arithmetic:

(S o T) o W = S o (T o W)

There is an identity transformation I that leaves unchanged any transformation it is combined wih: a circle rotation through 0 degrees.

T o I = I o T =T

Every transformation has an inverse: If T is any transformation, then there is anther transformation S such that the two combined give the identity:

T o S = S o T = I

These rules are true for the group of symmetry transformations of any object. These are called the axioms for a group G of transformations.

If G is the set of whole numbers. This works for o = addition or multiplication. P. 102

Three: When Computers Fail

The P vs NP Problem

This problem has to do with how efficiently computers can perform certain kinds of tasks.

In the 1930s, before computers came into importance as computing machines, some mtns were interested in "computability": what is computation and which functions can be computed. In the 1900s, David Hilbert was impressed with the success of the axiomatic method for doing math. This says that you begin any branch of math by formulating a set of basic assumptions – 'axioms"- and then generate all the facts in that branch by deducing them logically from the axioms. Thus "truth" reduces to "provable by the axioms". The Greek Mtn Thales first put forward this view around 700 BC, and formed the basis for Greek math. Euclid used this method in his book on geometry, *Elements,* around 350 BC. Hundreds of years of debate followed concerning the truth of one of his postulates; the so called Parallel Postulate: given a straight line and one point not on that line, you can draw exactly one straight line through that point that is parallel to the given line. This led to development of various non-Euclidian geometries. Euclid missed several but crucial assumptions you need to do geometry, and it was not until Hilbert took a closer look that a complete set of axioms was written down. Hilbert proposed that the same axiomatic process should be used to review all other branches of math. The search for axioms for various branches of math became known as the Hilbert Program. P. 106-107.

However, in 1931, Hurt Godel rocked the math world with his discovery that the axiomatic process does not work. He proved that in any part of math that includes elementary arithmetic, no matter how many axioms you write down, there will always be some true statements that cannot be proved from those axioms. This result, known as the Godel Incompleteness Theorem, destroyed the Hilbert Program. Building on Godel's work, other mtns started to investigate the concept of "computability". Theorems proved by Stephen Kleene, Alan Touring, and others established long before computers existed, the theoretical possibility of building programmable computers. P. 108-109

Michael Rabin's work in complexity theory. CT analyzes computational processes to see how efficiently they can be carried out. In 1970, Stephen Cook published his paper *The Complexity of Theorem Proving Procedures* in which he introduced a new theoretical concept, NP completeness. Problems that are NP Complete cannot be solved effectively by computer. One example is the Traveling Salesman Problem, where the shortest route between a number of random locations is desired. The only known way to solve this problem is to compute all possible arrangements of routes and then select the shortest. For small numbers of locations, this can be done by hand, but the difference between shortest and longest is not that much. As the number of locations increases, the number of possible combinations of routes increases exponentially, making it not a problem to be effectively solved by a computer.

Theoretical mtns asked the question: is it possible to prove that some problems simply cannot be solved effectively on a computer?

It turns out that computers can effectively solve problems involving a time complexity function that is a polynomial: the number of steps is CN^K

Where C is a constant, N is the number of data points, and k is the number of sub-steps.

Exponential time processes are of the form 2^N

Where N is the number of data points.

What makes certain problems all but impossible to compute may not be complexity, but the sheer number of simple calculations that need to be made. To distinguish such processes from those that were genuinely complex, another classification was made: nondeterministic polynomial time processes, or NP. The rest of this problem is too theoretical for me to have patience with.

Four:

Making Waves:

The Navier Stokes Equations. P. 131

In the early 1700s, the Swiss mtn Daniel Bernoulli showed how to use calculus to analyze the way fluids move when subjected to various forces. [this led to the Bernoulli equation, good for incompressible non-viscous flow, which states that Pressure + ½ density * (velocity of flow squared) is a constant. This equation is important in airplane aerodynamics.

Later, Leonhard Euler formulated a set of equations whose solution describes precisely the motion of a hypothetical viscosity free fluid. He was unable to solve these equations

In 1822, Navier amended Euler's equations to cover the more realistic case of a fluid having viscosity. Although his reasoning was flawed, his equations were correct. A correct derivation was obtained a few years later by George Stokes.

However, no one has been able to find a general solution the NS equations. We do not know if there is a math solution.

Isaac Newton in England and Gottfried Leibnitz in Germany developed differential calculus at about the same time. Leibnitz derivative notation dy/dt was more functional than Newton's "fluxions", and became widely used.

Author explains the derivative

Newton and Leibnitz' derivative looked at the infinitesimal motion of solid particles. Bernoulli's approach had to account for two infinitesimal phenomena: the infinitesimal motion of a single particle of fluid through space, and the infinitesimal motion of this particle of fluid (a streamline) with respect to the motion of another streamline. The key idea he used in his 1738 book *Hydrodynamics* was to take the solution to be a vector field. His work showed that when a fluid flows over a surface, the pressure the fluid exerts on the surface is reduced with increasing fluid velocity over the surface.

Special cases of the NS equations can be solved.

Author derives NS equations.

The equations in xyz coordinates through time are three complicated equations in x,y,z, and t.

Using vector calculus with a single vector in xyz space results in a much simplified equation using 4 operations of vector calculus.

Five

The Math of Smooth Behavior

The Poincare Conjecture p. 157

Poincare came close to beating Einstein in the development of the theory of relatively. Father of algebraic topology. Although maps and the math of networks are important, the study of topology was driven by a struggle to understand why differential calculus worked. P. 165