Math in 100 Key Breakthroughs
Richard Elwes
Quercus
2013
416 pages

This book is excellent in terms of content, revealing numerous insights. However, it also contains a number of typos or errors.

**Counting p. 8**
Experiments show that honeybees can reason abstractly, mentally associating different patterns which contain the same number of elements, up to 4. P. 9

New Zealand robins have shown their ability to distinguish between numbers (of objects) as high as 12. Baby chicks have the ability to watch carefully, remember the results, and have the intelligence to compare numbers in groups, to chose the group with the most numbers of baby chicks. P. 11

**Irrational Numbers p. 28**

Numbers that cannot be expressed as fractions. P. 29

**Platonic Solids p. 37**

Theaetetus' theorem proved that there are only 5 "platonic Solids", symmetrical shapes built from straight lines and flat faces. They are the tetrahedron, octahedron, dodecahedron, and icosahedron, made of symmetric triangles, and the cube, made of 4 squares.

**Conic Sections p. 56**
Conic sections in nature: Kepler showed that the planets orbits are not circles, but ellipses. Projectiles follow parabolic paths. The hyperbola is the rarest conic section found in nature. P. 59

**Trig p. 60**
Trig has always been associated with astronomy and technology; ie; positions. In 1400, the Indian Madhava discovered that expression of the sine of an angle required an infinite series:

(This expression is related to the factorial gamma function
See https://ocw.mit.edu/courses/mathematics/18-104-seminar-in-analysis-applications-to-number-theory-fall-2006/projects/chan.pdf )

The sine and other trig functions are called transcendental functions, because they require infinite series to express exactly. (e and pi are also transcendental numbers)

Trig functions have outgrown their humble roots in triangle geometry to become critical in complex analysis (p 154) and abstract waveforms (p. 169)

**Prime Numbers p. 49**
Although prime numbers occur infrequently in nature, they do appear. In 2001, the mathematical biologist Glenn Webb found that the 13 and 17 year life cycles of certain cicada species help them avoid resonances with the life cycles of their predators. Primes can increase to infinity. Euler wrote that Goldbach's conjecture, that every even number is the sum of two primes, as certain, though he could not prove it.

**Perfect Numbers p. 64**

Its factors summed = the number. Example: 1X2x3 = 1+2+3 =6. These numbers describe many of the symmetries of patterns and the natural world, eg: hex snowflake. Prime numbers of the form $2^p - 1$, where p is a prime number are known as Mersenne primes, or M. Euclid proved that a manipulation of a Mersenne prime will always be a perfect number. P. 66

**Algebra p. 84**

Bahman Kalantari of Rutgers university realized that the solutions to polynomial equations produced beautiful pictures, a technique he calls polynomiography p. 85.

**The Fibonacci Sequence p. 92**

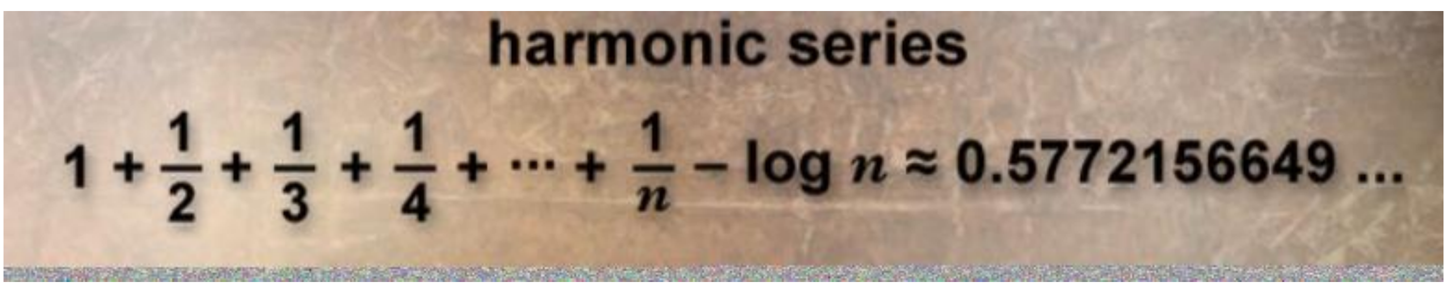
The Golden Section, studied by the Pythagoreans, and the Fibonacci sequence, first thought of by the Indian literary theorist Pingala, are closely related.

Phidias, the 5th century BC artist, sculptor, and architect, is traditionally recognized as the first to recognize the aesthetic possibilities of the golden section, defined by the ratio of lengths of a 5 sided star, or pentagram. Also by a square golden rectangle.
The Fibonacci sequence is formed such that each subsequent number is produced by adding the previous two.
This series grows without bound. The fractions of the successive members of the FS converge on the golden section, Phi.
This Phi can be used to construct any Fibonacci number without going through the entire sequence. This property was discovered by Jacques Binet in 1843.
P. 93 f.

**Harmonic Series p. 96**

In the 14th century, Nicole Oresme investigated the addition of infinitely many numbers.
A sum of numbers may be divergent or convergent, or may tend to a constant. Oresme asked what happens to a series of numbers that begins 1+ ½ + 1/3 + ¼ + 1/5 + …. ? this is called the harmonic series because these fractions are central to music theory. (physics) Although it looks like it might converge, Oresme showed that the harmonic series grows without limit, even though it may take many increments to do so. Euler showed that

Euler found that the harmonic series up to value n approaches ln (n) and the error is:




From Bressoud, The Queen of the Sciences: A history of Mathematics. The Teaching Company

the reciprocal of the prime number series also diverges: 1 + ½ + 1/3+ + 1/5 + 1/7 …, but that the square series converges: 1 + 1/2x2 + 1/3X3 + 1/4x4 … to a value of pixpi/6. These techniques developed into the modern subject called math analysis. The Riemann Zeta function is a generalization of the series of Oresme and Euler. P. 97 f.

The shape of the harmonic series curve describes a parabola, and the area under this curve

**Cubic and quartic equation solving p. 100**

Quadratic equation ($x^2$) simple to solve; cubic and quartic ($x^3$, $x^4$) harder. P. 101 f.

**Complex Numbers p. 104**

To describe quantum effects at the atomic lever, complex math is required. Euler probed the possibilities of complex numbers, and discovered that trig functions take on a dramatically new appearance in the complex domain. from that he developed his most famous equation: the Euler equation: $e^{i\Theta} + 1 = 0$

Around 1800, Caspar Wessel and Robert Argand proposed a geometrical interpretation of complex numbers: the x axis is "real", the y axis is "imaginary". P. 104.

**Logarithms p. 108**

John Napier discovered a way to turn difficult multiplication into simple addition: to multiply 2 numbers together; say 4587 X 1962, take the artificial number of the first, and add it to the artificial number of the second:
$10^{\text{Log10 } 4587 + \log10\ 1962}$
Today we understand that logarithms are the opposite of powers, which in turn are repeated multiplication.
2 to the power 3 is $2^3$, = 2 x 2 x 2 = 8 means: log of 8 to the base 2 is 3, or $\log_2 8 = 3$
Although the harmonic series diverges, the series 1 – ½ + 1/3 -1/4 + 1/5 does converge to about 0.693247 = ln2
ln2= log2, where e = about 2.71828, a very important number. P. 108

the equation y= 1/x is the reciprocal of x. its plot forms the hyperbola. The natural logarithm appears as the area beneath this curve. This is because the natural log is the inverse of the exponential function. It follows that the differential of the natural log y= ln(x) is the reciprocal y=1/x. p. 111.

**Kepler's Laws: p. 121**

**Differential Geometry p. 137**

A chain held at 2 points drooping in the middle is a curve called a catenary. A parabola is a reasonably good fit, but it is not a parabola. Johann Bernoulli. Tautochrone (same time) and brachistochrone (fastest time) for a physical object moving on a curved surface both turned out to be cycloids.

**Polar Coordinates p. 141**

What is the difference between an Archimedean and logarithmic spiral? Archimedean: turns of spiral same distance apart; logarithmic: spirals further apart as cycle outward. It is self-similar; meaning it looks the same no matter what distance you are from it, like fractals. Logarithmic spirals are ubiquitous in nature. Storm formations, spiral galaxies, nautilus shells and the flight paths of certain animals.

Archimedean and logarithmic spiral represented as $r = \Theta$ and $r = e^{\Theta}$ respectively. They are expressed much more simply in these polar coordinates, rather than Cartesian or rectangular coordinates.

**Normal Distribution p. 145**

If flip a coin x number of times, as x increases, the proportion of heads to tails will approach 0.0. this is known as the law of large numbers. Central to this theory is the normal distribution. The normal distribution occurs in a wide variety of physical situations. Central limit theorem: not explained.

**Exponentiation p. 153**

Power series are the result of adding up all the powers of the same number;

Eg: $1 + x + x^2 + x^3 + \ldots$

It was learned that the sine and cosine could be represented in terms of power series. Then Brook Taylor proved that most, or even all important math functions can be expressed as a suitably power series. Euler found an expression for the exponential function:

$e^x = 1 + x + x^2/2 + x^3/3x2 + \ldots$

the best known instance of this series is when x = 1, which gives e at about 1.7183. Euler also noted that the power series for $e^x$ looked similar to the power series for sin x and cos x, and from this he derived the Euler equation.
$e^{i\Theta} + 1 = 0$

**Euler Characteristic p. 157**

Euler found a remarkable relationship between the number of faces, edges, and corners in a polyhedron:

# vertices - # edges + # faces for any polyhedron will always be 2. This is true even if surface not flat nor edges not straight. This is also true of a sphere. This was one of the first great theorems of topology.

If a polyhedron has 0 holes, 1 hole, or 2 holes, it has a Euler characteristic of a sphere, torus, or double torus.

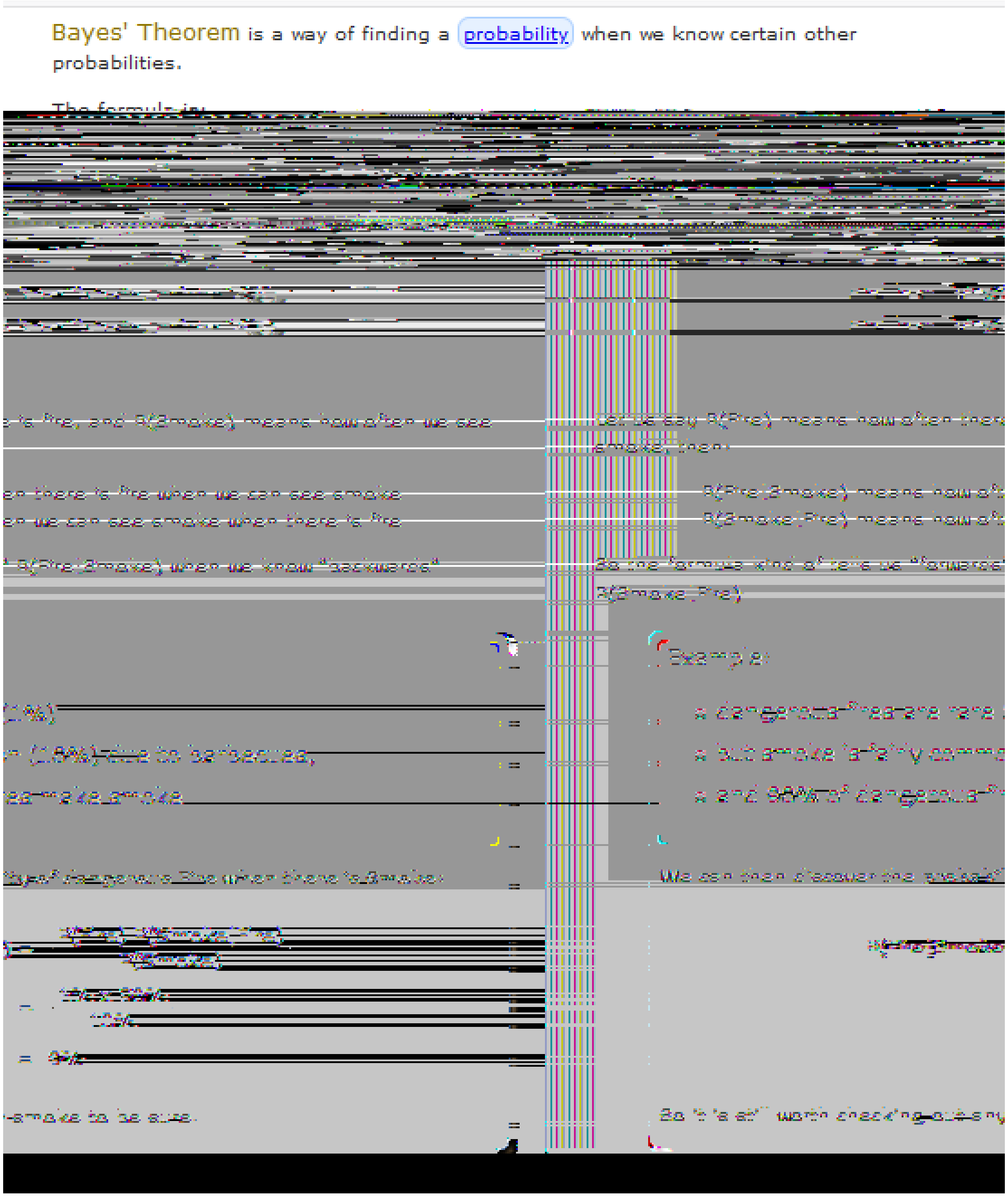
Poincare lifted this idea into higher dimensions. Just as a polyhedron is built from vertices, edges, and faces, so a higher dimensional polytope is built from cells of lower dimension. (p. 201). This will all lead to algebraic topology, in which structures inside a shape can be added or subtracted from each other, producing subtle algebraic objects. P. 159.

**Conditional Probability p. 161**

Bayes' innovation in probability was to allow the probability of multiple events to influence one another. Conditional probability and Bayes' theorem have become the foundation for the modern study of uncertainty. They are especially important in Markov processes (p. 245)

P(A|B) Probability of A given B
P(B|A) Probability of B given A
P(B) probability of B
P(A) probability of A

Bayes' theorem: P(A|B) X P(B) = P(B|A) X P(A) ?????? error in book


Bayes' Theorem is a way of finding a probability when we know certain other probabilities.

**Fundamental Theorem of Algebra p. 165**

In 1797, C. F. Gauss proved that complex numbers are enough to solve any equation built from real numbers. His proof had a few issues. In 1806 Robert Argand addressed the issues and proved that every equation built from complex numbers must already have solutions among the complex numbers.

Fourier Analysis p. 169

Fourier's original motivation was the study of heat transmission. Technically heat is a flow rather than a wave since its patterns do not repeat as in sound and light. However, the techniques he used for heat are equally applicable to waves. Fourier analysis underlies mobile phone technologies, radio communication, speech recognition, prime numbers and quantum mechanics.

**The Unsolvability of the Quintic p. 177**

In 1820, Niels Abel showed that although the Fundamental theorem of algebra guarantees a solution, the quantic was unsolvable using traditional math tools. P. 178

Evariste Galois work set the stage for what would become group theory, which would change the face of algebra over the 20th century. He noticed that the solutions of equations, like geometrical objects, have a characteristic called symmetry. Just as rotating a square by 90 degrees leaves its overall appearance unchanged, so the solutions of equations could be interchanged in such a way that its overall appearance remains the same.

His insight was that to truly understand an equation means understanding its symmetry. His theorem states that an equation is only solvable if and only if its group symmetries are solvable. The group symmetries for the quantic is unsolvable.

**Navier-Stokes Equations p. 181**

The movement of viscous flow can be described by the Navier Stokes equation.

Euler's approach to the mechanics of fluids was to watch what happened to Newtonian/Leibnitz mechanics of solids as the size of solid particles tended to zero. This required a new level of sophistication in applied calculus. However, for all Euler's technical brilliance, his equations failed to take viscosity into account. In 1822, Navier introduced a correction into Euler's equations which took into account viscosity, but he didn't get it right. 20 years later, Stokes got the corrected equations right. Since then, no one has been able to provide any satisfactory solution to the equations. Although every real fluid flow should satisfy them, all fluid flows that mathematicians have been able to model break down at some point. It may be that no solutions exist, but this would be unlikely, given the evidence of numerous computer simulations suggesting the NS equations are excellent at modeling reality.

This seems like a contradictory argument; it has not been explained properly. It seems to be saying that although the NS equations are excellent at modeling reality, they do not quite model reality.

**Curvature p. 185**

Gauss was able to produce a number quantifying surface curvature at a point. Gaussian curvature is positive on a shape like a sphere, but negative on a saddle. On a cylinder, Gaussian curvature is zero. Surfaces of zero curvature are exactly those that can be unrolled to lie flat.

Curvature is a local phenomenon, whereas topology is global. Yet Gauss discovered a very profound connection between these two approaches to shape analysis. Using integral calculus, he was able to integrate the curvature over an entire surface to arrive as a single number. This number was resistant to topological morphing. Even though local curvature morphed, this total curvature remained constant. This number was equivalent to the Euler characteristic times 2 π. Pierre Bonnet rediscovered this relationship, which became known as the Gauss-Bonnet theorem, which plays a role in the General Theory of Relativity.

**Hyperbolic Geometry p. 189**

Euclid's parallel postulate states that if you have a straight line, and a point away from it, exactly one line parallel to the original line may be drawn through that point. Gauss, Lobachevsky, and Bolyai each discovered a new type of geometry in which Euclid's first 4 laws hold, but the parallel postulate does not. It was called hyperbolic geometry. In this, many possible lines could be drawn through a point all parallel to the original.

Riemann, one of Gauss' students, realized that the difference between Euclidian and hyperbolic geometry could be understood in terms of curvature. The Euclidian surface is flat with zero curvature, while hyperbolic space has negative curvature. One consequence is that three angles of a hyperbolic triangle ass up to less than 180 degrees. On a sphere, with positive curvature, the three angles add up to more than 180 degrees. The age of Euclidian geometry ended, and the age of Riemannian geometry began. Riemannian and hyperbolic geometry would play central roles in our understanding of the universe, including the GTR.

**Constructible Numbers, p. 193**

Pierre Wantzel found that everything constructible by ruler and compass must result in some combination of addition, subtraction, multiplication, division, and square roots. Just as classical ruler and compass problems yield constructible numbers, so numbers producible by folding are origami numbers. It turns out that every constructible number is origami, but not vice versa.

**Transcendental Numbers p. 197**

In 1844, Joseph Liouville found as new type of number. These numbers can not be described in terms of whole numbers. Square roots of numbers are irrational, but are described in terms of whole numbers. Were these numbers just curiosities?

It was proven that both e and π, ubiquitous in physics and math were both transcendental.

In 1874, Georg Cantor showed that there are different levels of infinity, some bigger than others. He also showed that the infinity of transcendental numbers is larger than that of non-transcendental or algebraic numbers. This means that there are infinitely more transcendental numbers than all of the usual numbers that mathematicians deal with.

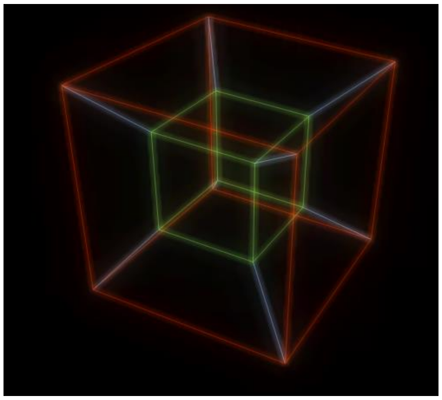
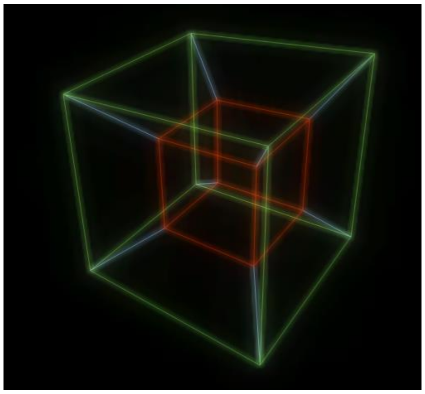
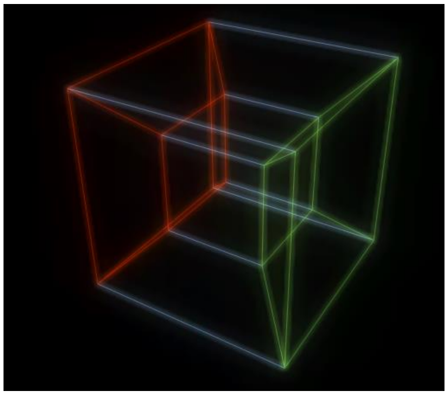
In the early 20$^{th}$ century, it became clear that the number e is not just transcendental, but is the key to the entire phenomenon.

It is not known even today what $e^e$ or e + π means.

**Polytopes p. 2201**

In the Euclidian plane, two members, 1, 2 represent a point; 3 numbers, 1, 2, 3, represent a point in 3-D space. In the same way, 4-D geometry can be considered as the study of quadruples of numbers such as 1, 2, 3, 4 in 4-D space. All the usual geometric values such as angle, length, and volume generalize into this dimension.

See [https://observablehq.com/@toja/4d-hypercube](https://observablehq.com/@toja/4d-hypercube)

**Riemann's Zeta Function p. 205**

Using Riemann's Zeta Function, it is possible to produce an explicit equation for the number of primes below any limit. The critical information for the primes consisted of inputs for which the output was zero. Riemann hypothesized that non negative zeros all lay on a vertical line at x=1/2. This was called the Riemann hypothesis, which has never been proven.  It was proved that all of these zeros must lie inside a critical strip surrounding the critical line at x=1/2. This is known as the prime number theorem.

There is no apparent connection here between prime numbers and physics.

**Jordan Curve Theorem p. 209**

Basically says any loop in 2-D space, or any bubble in 3-D space, has an inside and an outside.  Alexander's Horned Sphere is an example. It has an inside and outside, but is infinitely complex.

Classification of surfaces p. 213

Mobius strip has only one side; Klein bottle: a closed surface with no edges and only one side; formed by fusing the edges of 2 mobius strips. The ones in 3-D have a flaw where the surface cuts through itself.  In a 4-D world, a flawless Klein bottle could be formed.

**Cardinal Numbers p. 217**

Georg Cantor perceived that infinity is not something magical; it is amenable to analysis. He developed set theory, and was interested in comparing the different sizes of sets. It was noted that if one removes the set  all  odd numbers from the set of all numbers, both sets remain the same size, and infinite, because each odd number can be paired with an even number. Another infinite set is the collection of all real numbers, each of which require an infinite number of digits, as for example the real number π. This set is infinitely larger than the set of whole numbers, so real numbers are a power set of whole numbers.

**Wallpaper Groups p. 221**

**Digital Geometry p. 225**

**Russel's Paradox p. 229**

**Special Relativity p. 233**

Galilean relativity: Galileo stated that there is no way to distinguish the physics in a room moving at constant speed from one that it stationary.  This is consistent with Newton's law of gravity, but applies only at slow speeds

Lorentz transformations show odd differences between two observers traveling at speeds measurable relative to the speed of light.

Minkowski space mathematically models a universe in which the STR holds, in 4 dimensions. Lorentz transformations represent symmetries of this structure, just as rotation of a square is a symmetry of that shape.

**Three Body Problem p. 237**

Chaos Poincare first investigated this problem. In 1912, astronomer Karl Sundman produced a series solution, meaning that an infinite series of terms is required.  Because the series converges very slowly, too many terms are required to get any useful information from it.

**Waring's Problem p. 241**

Primarily a math problem with no physical meaning.

**Markov Processes p. 245.**

First formulated by Andrei around 1910, these processes model situations in which there is short term uncertainty, but long term prediction. In a perfect grid, where a random walker can choose any of 4 directions at an intersection, the walker has a 100 % chance of reaching any specified point, even on an infinite grid. These processes appear throughout science, modeling chemical reactions, and radioactive decay. In later years, they have been useful in modeling situations that are not truly random, but whose behavior depends on so many variables that it is impossible to predict, like fluctuations in the stock market. Claude Shannon even modeled human language as a Markov process, in which one symbol follows another with a given probability.

**General Relativity. 249**

The Einstein field equation involves Gregorio Ricci-Curbastro and Tullio Levi-Civita's curvature tensor calculus equation, which is a fundamental measure of the curvature of space. In the context of space-time this is modified to give the Einstein tensor, denoted G. Einstein's insight was that the curvature of space-time is determined by the amount of matter or energy present, expressed by the stress-energy tensor, T, so that G= T. to find a geometric description of the shape of space-time, this challenging equation must be solved.

Geodesics are the answer to the question of what is the shortest path between two points. On a flat plane, a straight line is the shortest path, while on a sphere, geodesics are portions of a great circle generated by a plane through the center of the sphere cutting the sphere surface. In GTR, the geodesics of space-time describes the paths of free falling objects, so it is gravitational free fall, rather than motion with constant velocity, that is fundamental.

Black holes: Covered in other references.

**Fractals p. 253. Covered in other references.**

Gaston Julia was the first to illustrate how complex shapes may emerge from iterating simple procedures at different points. The results gave identical patterns at all scales.

Mendlebrot gave these complex shape the name "fractals" and made them popular in the book *The Fractal Geometry of Nature.*

Many aspects of the physical world demonstrate quasi fractal patterns, from snowflakes, to ferns, to river deltas and tree branches and vegetables.

**Abstract Algebra p. 257**

Successive generations of mathematicians realized that ordinary whole numbers are far deeper than at first it seemed, as is recognized by work such as the Riemann hypothesis and Godel's Incompleteness Theorem. Studying these relationships lead to structures which are more abstract, but in many ways logically simpler than the system of whole numbers. Architectural designs such as the Guggenheim museum in Bilboa Spain rely on algebraic geometry described by computer (CAD). Envisioning of Non-Euclidian geometry as well as higher dimensional spaces has resulted in more complex shapes.

Emmy Noether undertook analyzed the notion of rings as the rules of addition, subtraction, and multiplication as applied not just to numbers, but more generalized entities. She saw how rings were constructed from subsystems called "ideals", which illuminated math topics such as modular arithmetic. Matrices are another type of ring, which can be added, subtracted, and multiplied, but since matrices A X B not equal B x A, they are non-commutative.

Noether's investigation of pure algebra had a great impact on geometry. Descartes' cartesian coordinates linked algebra and geometry; equation and shape. This algebraic approach to geometry began to flourish through the work of Oscar Zariski. As a result, it became possible to make meaningful geometric statements, even when there was no obvious physical interpretation. Abstract structures called “varieties” and “schemes” have become the vocabulary of 21st century geometers.

**Knots p. 261**

In 1923, James Waddell Alexander found a way to describe knots using algebraic techniques. To every knot was assigned a polynomial. These Alexander polynomials provided a reliable way to determine if two knots were truly different. In 1984, Vaughn Jones found a better alternative algebraic description of knots. Jones’ breakthrough soon found scientific Application: Ultimately, knot theory is about the interaction between the geometries of different dimensions. The knot is a one dimensional object sitting in 3-D space. Similar considerations arise in quantum field theory, making knot tools indispensable. The Jones polynomial is also used by biochemists to understand the way that enzymes with cells manipulate strands of DNA.

**Quantum Mechanics p. 265**

In the early 19th century, Thomas Young found that a beam of light projected on a screen through a pair of slits resulted in an interference pattern. This seemed to demonstrated that light was a wave. In the 20th century, the photo-electric effect, in which light beam falling on metal releases electrons, suggested light had a particle nature. Einstein explained this anomaly by thinking of light as energy packets of a fixed size, which he called photons.

J. C. Maxwell was aware of the results of the double slit experiment, as well as the photoelectric effect and Einstein’s interpretation, suggested the particle as well as the wave nature of light. However, he cautioned against thinking of the photon as either wave or particle.

Later, the double slit experiment was repeated, firing only one photon at a time. Gradually, an interference pattern appeared. The implication was that even individual photons exhibit both particle and wave characteristics. This resulting interference pattern could be interpreted as the probability of a single photon arriving at a particular location on the screen. Classical probability distributions however, like the normal distribution, could not be used because they cannot interfere with one another, as waves do. A better description required the development of what we now call “quantum wave functions”, which are expressed in complex numbers, allowing both positive and negative interference.

Image of sin and cosine interfering

Schrodinger’s wave equation elegantly describes the way in which waves propagate through space as a function of their energy.

**Quantum Field Theory p. 269**

QM does not describe how particles interact, so it had to be imbedded in a larger theory: quantum field theory. QM seemed to be incompatible with STR and GTR.

Paul Dirac tried to develop a relativistic form of the fundamental theory of QM, the Schrodinger wave equation. The resulting Dirac equation became the basis for QFT.

$E = MC^2$ of STR holds that particles spring into and out of existence. Dirac suggested that electrons should be viewed as excitations of something more fundamental: an electron field, just as photons are viewed as excitations of the electromagnetic field. Dirac’s equation allowed for two solutions, both positive as well as negative electrons. The positron was discovered in 1932 by Carl Anderson. Scientific belief is that for every particle there is an antiparticle. An enduring mystery of the universe is why matter is so much more abundant than antimatter.

Dirac’s work did not take into account the physical processes through which particles interact. In the 1940s, the interaction between matter (actually electrons) and electromagnetic radiation (EM forces) was nailed down by Richard Feynman, Sin-Itiro Tomanga, and Julian Schwinger. The resulting theory of quantum electro dynamics (QED) has achieved stunningly accurate results for the interaction of electrons. Over the next several decades, theoretical physicists worked to develop quantum field theories for the weak and strong nuclear forces. These three quantum field theories formed the basis of the SMPP, which still is not consistent with the force of gravity.

Renormalization:
In the early days of QED, “it was noticed that the weak nuclear force” Should Be: “electromagnetic force” when viewed at the smallest scales, was expressed by divergent terms going to infinity. These terms were systematically eliminated by a process called renormalization, assuming the infinities cancelled out. This technique has remained controversial among some physicists, including Richard Feynman. Controversy also exists in the so-called Yang-Mills problem.

**Ramsey Theory p. 273.**

In 1930, Frank Ramsey published a paper probing the boundary between order and disorder. “The exact numbers which arise from the subject remain deeply mysterious.” His theory showed that one can always find a very simple highly ordered structure lurking within an apparently disordered structure.

**Godel’s Incompleteness theorem P. 277 also covered in other references.**

Central to Godel’s work was the phenomena of self-reference. Other theorists had been using logic to describe numbers. Godel realized that numbers could be used to describe logic. He revealed that any logic system powerful enough to describe arithmetic must also be able to describe itself. He was able to encode in numbers the statement “This statement has no proof”. If this statement is false, this meant the logical system could provide a proof for an untrue statement. So it had to be true. With that, Godel found an example of a true but unprovable statement . from this, several new branches of math evolved.

**Turing Machines p. 281**

**Numerical Analysis p. 285**

Although math is an exact science, the tools of math are applied across countless areas of science, technology, and engineering, where total precision is unavailable or even unnecessary. In order to be useful in the real physical world, a range of methods for approximate solutions has evolved. This has been called numerical analysis.

Those of us who studied math in school are used to questions where [there] is a single exactly correct answer. In many cases, this is not possible. Approximations of square root of 2 and π are early examples. An early breakthrough was the Newton-Raphson method: first assume a solution, then use the result iteratively to arrive at better approximations. Numerical techniques can be tricky, typically they require deep theoretical justification.