Physics and Geometry

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Abstract

Geometrical ideas have played a crucial role in the development of Physics. Einstein's General Theory of Relativity is a classic example of a theory which is fully geometric. It can be shown also that Gauge Field Theories have a deep geometric meaning, where the potentials of the gauge field play the role of a *connection* over some fibre bundles, and the gauge field associated with this potential is simply the curvature of the connection.

1 Introduction

Einstein's biggest dream was to geometrize all the fundamental forces of nature. The idea of geometrization is reflected in his General Theory of Relativity (GTR) where gravitation is described as a purely geometric effect through his famous equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is a geometric object describing the geometry of space-time and $T_{\mu\nu}$ is the energy-momentum tensor.

One of the consequences of the space-time being curved is the introduction of another derivative, the covariant derivative, which reduces to the ordinary derivative in the limit of a flat space-time

$$\partial_\mu X^\nu \longrightarrow \nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\gamma\mu} X^\gamma, \tag{2}$$

where $\Gamma^\nu_{\gamma\mu}$ is an object called *Riemann connection* and it plays the role of the gravitational forces in the GTR.

In the subsequent years after constructing the GTR, Einstein tried to construct a unified field theory for all the forces of nature. In this theory matter and forces would be manifestations of geometric objects, but he failed to achieve such unification for more than 30 years.

2 Geometry of Gauge Fields

The idea of gauging was first proposed by Weyl while he was trying to incorporate electromagnetism into geometry using space-time dependent (local) *scale* transformations (see [2]). Namely, at a neighbouring point the scale is changed from 1 to $1 + S_\mu\, dx^\mu$. Hence, a space-time dependent function will change to

$$f(x) \longrightarrow f + \big[(\partial_\mu + S_\mu) f\big]\, dx^\mu. \tag{3}$$

Weyl tried to derive electromagnetism by requiring the invariance under this transformation and by identifying the scale factor with the vector potential: $S_\mu \to A_\mu$. His initial attempt was not successful.

The idea of gauge transformations emerged again after the advent of modern quantum mechanics. In quantum mechanics, the wave function describing any physical system can be transformed according to

$$\psi \longrightarrow e^{i\alpha}\psi, \tag{4}$$

where α is an arbitrary constant phase. The fundamental equations of quantum mechanics are invariant under this transformation. For example, the Lagrangian of a free Dirac field

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi, \quad \not{\partial} = \gamma^\mu \partial_\mu \tag{5}$$

will remain the same under transformations (4).

Let's turn now to the case when the phase transformations (4) are local, i.e, space-time-dependent α. Our Lagrangian (5) will no longer be invariant. The presence of a space-time derivative in the Lagrangian will contribute a term which will spoil the gauge symmetry. This situation reminds us of the GTR where the transition from flat space-time to a curved one results in the introduction of a covariant derivative. This reasoning motivates us to introduce a covariant derivative (and a *gauge connection*) in our theory of gauge fields. The Lagrangian (5), when written in terms of this derivative, will remain invariant under local gauge transformations.

Before proceeding further, let's first study the structure of the transformations (4). In the local case they will have the form:

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x). \tag{6}$$

This (one-parameter) Abelian group of unitary transformations is called $U(1)$. The transformations represent rotations in the internal space of the fields. These rotations are space-time dependent, i.e., the rotation angle will change from one point to another in space-time. It is worth mentioning that these rotations have nothing to do with ordinary rotations in $\mathbb{R}^3$. We can imagine that at each point of our space-time there is a circle and rotations around these circles are represented by the transformations (6) (see [3]).

Now it is the time to introduce a very useful concept from differential geometry. From the above discussion we have a base manifold $\mathcal{M}^4$ (Minkowski space-time) together with a circle at each point (fibres $\mathcal{S}$). Both the base manifold together with the fibres form what is called a *fibre bundle*; we will call this fibre bundle $\mathcal{M}^4\mathcal{S}$. The covariant derivative is

$$D_\mu = \partial_\mu + iA_\mu, \tag{7}$$

where A_μ plays the role of a connection over the fibre bundle. The Lagrangian (5) in terms of the covariant derivative reads

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi, \quad \not{D} = \gamma^\mu D_\mu \tag{8}$$

which describes a Dirac field interacting with the electromagnetic field. To add a kinetic term describing the free electromagnetic field in this Lagrangian we can think about the curvature of the connection A_μ. In GTR the curvature of the connection $\Gamma^\gamma_{\nu\mu}$ is given by the Riemann tensor ([1])

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{9}$$

This suggests that we choose as a curvature of our gauge connection A_μ the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10}$$

The complete Lagrangian of a Dirac field interacting with an electromagnetic field can now be written as ([4])

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{11}$$

3 Conclusion

Our discussion so far considered only the simplest case of a one-parameter group $U(1)$ which is the gauge group of electromagnetism. The extension to include other gauge fields is done via Yang-Mills theory. The gauge potentials in this case will carry another index beside the space-time one. The gauge groups in the Yang-Mills theory are no longer one-parameter (hence, non-Abelian) and the fibre bundle structure will be more rich. It is worth mentioning that GTR is another example of a gauge theory where the gauge group is the group of all diffeomorphisms on the space-time manifold. The study of the connections between space-time symmetries and gauge symmetries may in the end lead to a unified theory for all the fundamental interactions of nature.

References

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